Self-referred approach to lacunarity

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This paper describes an approach to lacunarity which adopts the pattern under analysis as the reference for the sliding window procedure. The superiority of such a scheme with respect to more traditional methodologies, especially when dealing with finite-size objects, is established and illustrated through applications to diffusion limited aggregation pattern characterization. It is also shown that, given the enhanced accuracy and sensitivity of this scheme, the shape of the window becomes an important parameter, with advantage for circular windows.

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Several interesting natural and abstract phenomena and structures are characterized by intricate geometries whose properties can vary along space and/or time [1–5]. Chaotic dynamics, for instance, is known to be organized in terms of fractal attractors [6], which are characterized by self-similarity or self-affinity over spatial scales. Given that great part of the systems exhibiting particularly interesting behavior involves such complex geometrical organizations, it becomes necessary to have proper and effective measurements allowing the objective and meaningful quantification of specific geometrical features, such as regularity, density, self-similarity, and translational invariance. One important point to be highlighted at the outset is the fact that such measurements are almost invariably *incomplete* or *degenerated*, as a consequence of the mapping from a higher dimensional space, where the structures are embedded, into a lower dimensional space. Therefore, while it is often unfeasible to incorporate all information into geometrical measurements, they must be capable of expressing the features of particular interest with respect to each specific application. For instance, the characterization of the distribution of voids in different sizes and shapes is a major factor to be considered while specifying the mechanical properties of a metal bar, for example, establishing an intrinsic relationship between topological or geometrical properties and physical strength that can be, to some extent, captured by its porosity value in case a parsimonious description is needed. By providing accurate and meaningful information about the specific geometrical properties of interest, suitable measurements of complex structures allow the construction of statistical models of the analyzed objects and the identification of prototypes, as well as the taxonomic organization of several types of patterns. Such possibilities are important not only for practical applications, but also for theoretical studies aimed at investigating critical phenomena and universality [7].

One of the best known measurements of complex structures is the *fractal dimension*, introduced by Mandelbrot, see [8]. Although several alternative definitions of such a measurement have been available for a long time (e.g., [9,10]), they all assume self-similar (or self-affine) symmetries while sharing the ability to quantify the spatial "complexity" of given patterns. Although powerful and widely used, the fractal dimension is inherently a degenerated feature, implying an infinite amount of distinct patterns to be mapped into the same fractal dimension. The concept of *lacunarity* [7,8,11] has been introduced and used as a means to complement the quantification of complex geometries provided by the fractal dimension. In particular, the lacunarity quantifies the degree of *translational invariance* of the analyzed objects, with low values of lacunarity indicating high levels of such an invariance. For instance, a pattern with uniform spatial distribution of voids will lead to small lacunarity values. One especially interesting property of the lacunarity, which is not shared by the traditional fractal dimension, concerns the fact that it is defined in terms of a *scale parameter*, namely the size of the sliding window which is used to analyze the patterns. As a consequence, a lacunarity function, rather than a single scalar value, is obtained which characterizes the translational invariance with respect to several spatial scales, providing enhanced information about the geometrical properties of the analyzed patterns. A particularly representative illustration of the potential of the combined use of the fractal dimension and lacunarity is related to the characterization of DLA structures which, by being organized around the initial "seed", tend to exhibit distinct geometrical properties around that seed and also at the DLA boundaries [12].

Despite the promising potential of the lacunarity as a measurement of complex patterns, some remaining intrinsic difficulties have constrained its applications. Indeed, while lacunarity and fractal dimension are often successfully considered for the characterization of infinite/periodical structures, the treatment of finite and isolated objects, implied by many relevant natural situations, has received relatively little attention in the literature. The key issue here is that, when dealing with finite-size objects, one is interested in their intrinsic geometrical properties, preferably in a way invariant to translations and rotations. By depending substantially on the placement and orientation of the finite object along the workspace, the traditional approach to lacunarity implies a high degree of arbitrariness, in the sense that completely different values can be obtained for the same object. The adoption of the object as a reference system for calculation of the lacunarity allied to the choice of circular windows, as adopted in the present article, completely avoid such a prob-

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lem, providing invariant characterization of finite-size objects.

The current work investigates the use of the analyzed pattern itself as the reference for the windowing procedure underlying lacunarity estimation. Although such an approach has been considered previously [11], the restricted conditions adopted for its validation (Cantor dust) implied its premature dismissal. An interesting informal interpretation of this approach is in understanding the structure of interest as being measured by an inhabitant of the object who, therefore, can only sample a circular region around each of its positions. We show in the following that this self-reference windowing system does allow a series of superior features, including enhanced objectivity, accuracy, and sensitivity, also implying the shape of the sliding window to become critical for proper operation. It is shown that such an alternative procedure allows the additional bonus of enhanced computational speed.

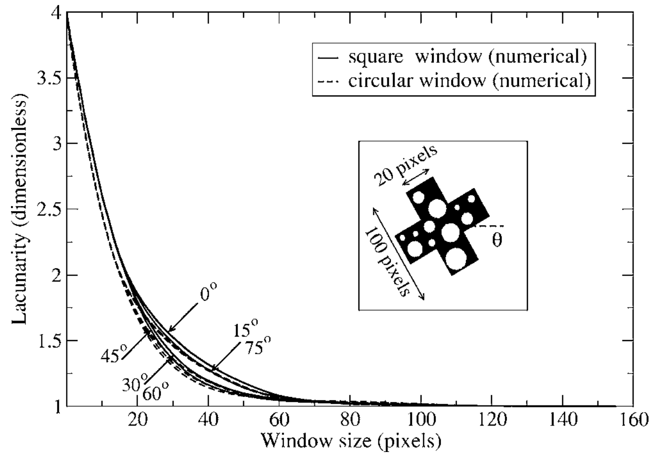

FIG. 1. The standard implementation of the lacunarity concept for finite objects characterization yields different lacunarity signatures depending on the rotation (by an angle θ) of the object under analysis shown in the inset.

I. METHODOLOGY

One of the most traditional approaches to estimate the lacunarity of a set of objects is known as the sliding-box algorithm, (e.g., [11]). Provided the set under analysis is mapped into an orthogonal lattice, henceforth called the *workspace*, a square window of side l is made to slide through the entire lattice while the number of pixels which falls inside it is determined. Let $n(s,l)$ be the number of such windows which contain s pixels and $N(l)$ be the total number of windows of size l. The probability of finding a box of size l with s pixels is given by $Q(s,l)=n(s,l)/N(l)$, and the lacunarity $\Lambda(l)$ of such a pixel distribution can be expressed as

$$\Lambda(l) = \frac{\sum s^2 Q(s,l)}{\left[\sum sQ(s,l)\right]^2} = \frac{\sigma^2(l)}{\mu^2(l)} + 1, \quad (1)$$

where μ and σ^2 are the mean and variance of $Q(s,l)$. Although popular, such a procedure involves some arbitrariness related to the difficulty of choosing the several involved parameters such as the position and size of the workspace and the shape of the sliding window.

Figure 1 shows the traditional lacunarity curves numerically obtained for rotations of the considered pattern, shown in the inset, considering both square and circular windows. We will adopt henceforth the term *window size*, which for a circular window refers to its radius l and for a square window corresponds to half of its side, *i.e.*, $L/2$. A substantial variation is observed for both square and circular windows. Figure 2 presents three lacunarity signatures obtained for three distinct relative sizes of the working space and object. It is clear from these curves that the choice of proportionality ratio between the workspace and the object size has great effect in defining the lacunarity values. The curves in Fig. 3 were obtained for a fixed working space size, but with the object (a cross shown in the inset) placed at different relative positions, given by the parameter Δ. A strong variation of the obtained lacunarity values was again observed, indicating arbitrariness also regarding the object position. Therefore, the large variations implied by the above arbitrary choices undermine the potential of the lacunarity as a sensitive measurement of the spatial distribution of the analyzed finite structures.

The arbitrariness identified previously can be completely removed by the use of the structure under analysis as the reference for placing the sliding windows. In other words, the window is placed at each of the points of that structure, eliminating the influence of the workspace, which can now be objectively defined by considering the maximum sliding-window size and the structure under analysis. The remaining parameters are, therefore, reduced to the shape of the sliding-window and the spatial-scale interval of the analysis (i.e., the range of window sizes), accounting for enhanced objectiveness of the whole approach. The self-referred approach to lacunarity estimation is illustrated in Fig. 4, including some (labeled a, b, c and d) of the many required positions of the sliding window. The total number of these windows $N(l)$ actually matches the area of the cross (i.e., the number of

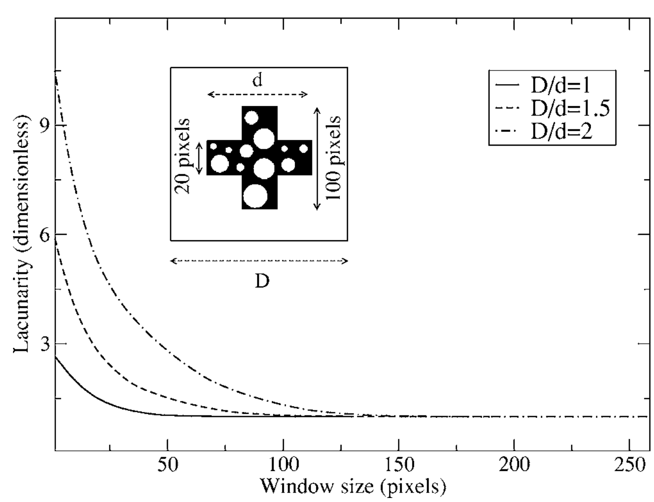

FIG. 2. The influence of the relative size of the object with respect to the workspace D/d on the values of the lacunarity, as measured by the standard approach for the object shown in the inset.

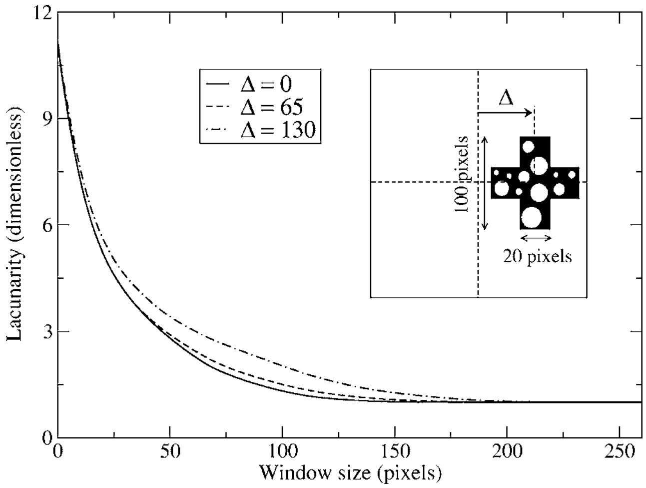

FIG. 3. The effect of translation by Δ on the lacunarity of the object shown in the inset, as calculated by the standard procedure.

elements or pixels in the cross, see Fig. 4). Note that the windows b and c overlap, i.e., there is a portion of the object which is covered by both windows, a situation which often occurs. A zoomed version of the window d is presented in the inset e, showing the exact portion of the cross which is covered by this window, i.e., the 198 elements inside the circle.

Consider the case of a simple object (a square), illustrated in gray in Fig. 5, for which it is possible to calculate analytically the self-referred lacunarity. For windows size r within the interval $[0,L/2]$, the functions $A_i^1(x,y,r)$, giving the area of the object enclosed by the sliding window as a function of

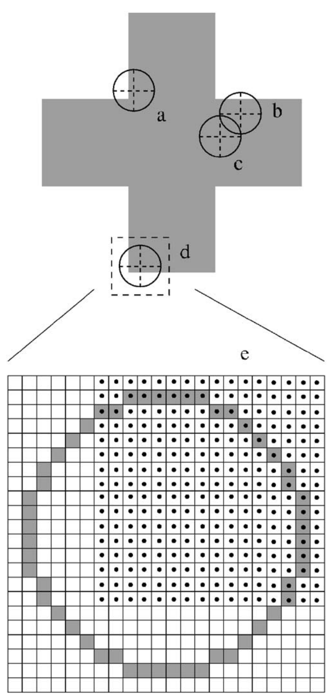

FIG. 4. The self-referred lacunarity method with circular sliding window for a simple object (a cross). One can see in the zoomed inset the part of the object, represented by black dots, which is being considered by window d.

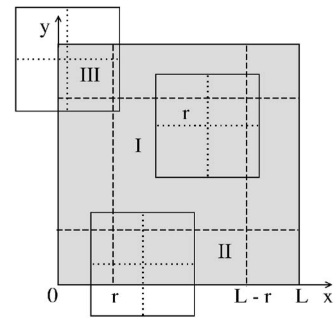

FIG. 5. A simple shape (a square, shown in gray) and the regions for the analytical calculation of the self-referred lacunarity for $r\in[0,L/2]$. The sliding windows in the region I fall completely within the object, while the windows in the cases of regions II and III always have a portion falling outside the object.

the window size, are given by the following equations:

$$A_I^1(x,y,r) = 4r^2,$$

$$A_{II}^1(x,y,r) = 2r^2 + 2ry,$$

$$A_{III}^1(x,y,r) = r^2 + [r + (L-y)]x + r(L-y).$$

The mean value of the area enclosed by the sliding window as a function of r is determined by the integration of the functions A_i^1 along their respective domains (see Fig. 5), divided by the total area of the object, given as

$$\mu_1(r) = \frac{\int_x \int_y \sum_i A_i^1(x,y,r)dx\,dy}{\int_x \int_y dx\,dy} = \frac{r^2(r-2L)^2}{L^2}.$$

The corresponding variance is calculated in a similar way.

The calculation of the area of the object inside a window of size r, with $r\in[L/2,L]$, implies the division of the object into specific regions which are analogous, although different, from those shown in Fig. 5 (see also [13]). The calculation of the mean and variance values is straightforward. The final expression for the whole interval self-referred lacunarity is given in the following:

$$\Lambda(r) = \begin{cases} \dfrac{4L^2(5r-6L)^2}{9(r-2L)^4}, & r\in[0,L/2]; \\ \dfrac{L^2(2r^3-6rL^2+L^3)^2}{9r^4(r-2L)^4}, & r\in[L/2,L]. \end{cases} \qquad (2)$$

Figure 6 shows a plot of the values obtained by using Eq. (2) for a square of size $L=100$ pixels.

In order to better illustrate the invariance of the self-referred lacunarity, it has been applied also for the cross-shaped object shown in the upper left portion of Fig. 7. Its self-referred lacunarity considering diverse rotation angles,

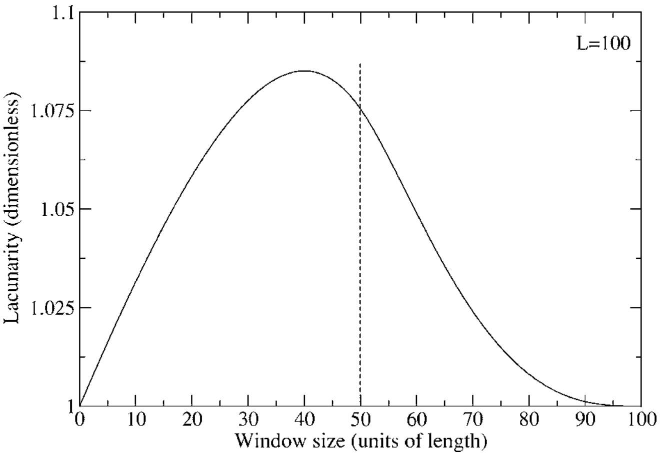

FIG. 6. The analytically calculated self-referred lacunarity curve for a square with size $L=100$ pixels.

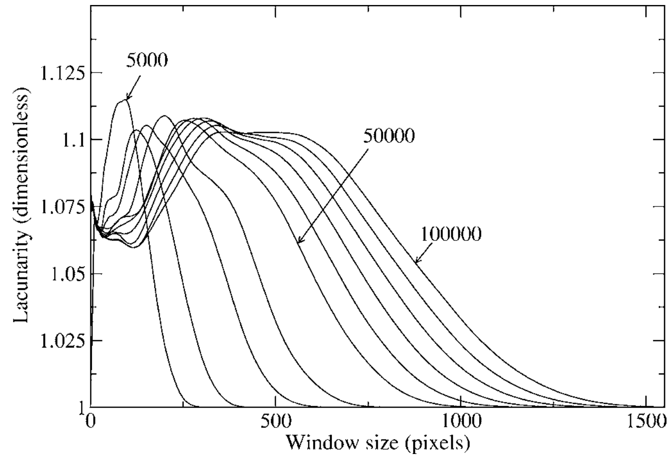

FIG. 8. The effect of the number of aggregated particles on the self-referred lacunarity value.

two sliding window shapes, as well as the respective analytical result, are presented in that same figure. The analytical result was obtained by decomposing the cross into several parts and solving, analytically, the sums implied by the definition in Eq. (1). The full details of such a calculation, that follows the same reasoning as described above for the square object, can be found in the supplementary material to be found in [13]. In the main graph we can easily spot two main groups: one associated with the square sliding window and another, more tightly grouped, associated with the circular window. Among the more widespread group, one can see a solid line representing the analytical calculation expected for the self-referred method, which matches precisely the numerically evaluated curves. The inset provides a zoomed

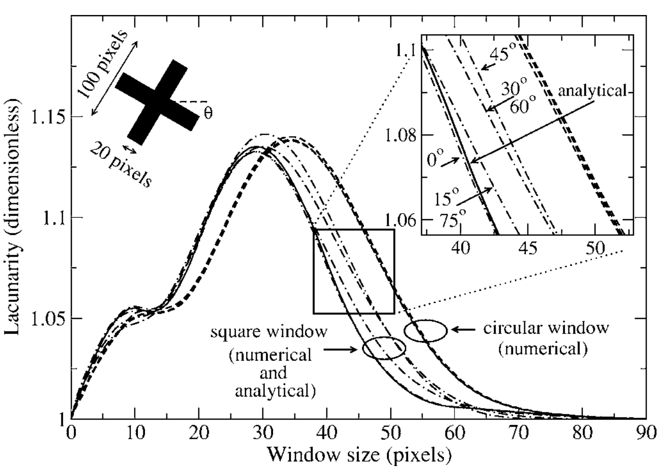

FIG. 7. Self-referred lacunarity signatures obtained for a cross considering square and circular windows, indicated by the arrows in the bottom of the figure. The simple cross was adopted in order to allow comparison with the analytical expected lacunarity, which is also shown in this figure. While the square windows with different orientations led to substantially different lacunarities, the circular windows were almost completely invariant to rotations, producing nearly undistinguishable lacunarity signatures. The same rotation angles (i.e. 0° , 15° , 30° , 45° , 60° , 75°) were used for both square and circular windows.

view of the variance implied by the use of a square sliding window, which is particularly critical if rotational invariance is required. Observe that the object (a cross) is shown without reference to the workspace because the results are completely independent of such a choice.

II. RESULTS

An important feature of many quasi-self-similar shapes is the existence of a descriptor, such as the fractal dimension, which can provide a characteristic signature for the shape regardless of the number of aggregated particles. The lacunarity represents one such a descriptor which has been proposed in order to complement the fractal characterization [2]. Figure 8 illustrates an interesting property of the self-referred lacunarity with respect to the standard procedure for DLA generation [14], corresponding to the fact that the peak value of the lacunarity curves obtained for DLAs with increasing number of particles tends to converge to a stable value. Such a value suggests itself as a possible measurement for characterizing the whole sequence of produced individual DLA shapes.

Another important issue is related to the sensitivity of both standard and new approach to small variations of the object. We consider this important perspective through an experiment where the object is perturbed by increasing Poisson noise. This was performed by distributing through the workspace black points with increasing densities λ_n (indicated in Figs. 9 and 10). The outcome of such a study is presented in Figs. 9 and 10. The traditional approach to lacunarity (for an object with the shape of a cross) is shown in Fig. 9 for several levels of noise quantified by the respective Poisson rates λ. Figure 10 shows the corresponding curves for the self-referred lacunarity for the respective degree of noise. The traditional lacunarity is characterized by maximum relative variation of 0.73 against 0.13 for the proposed approach. Such a result suggests that the self-referred method present enhanced robustness when compared to the traditional lacunarity.

In order to investigate the potential of the self-referred lacunarity approach for pattern discrimination, an experiment

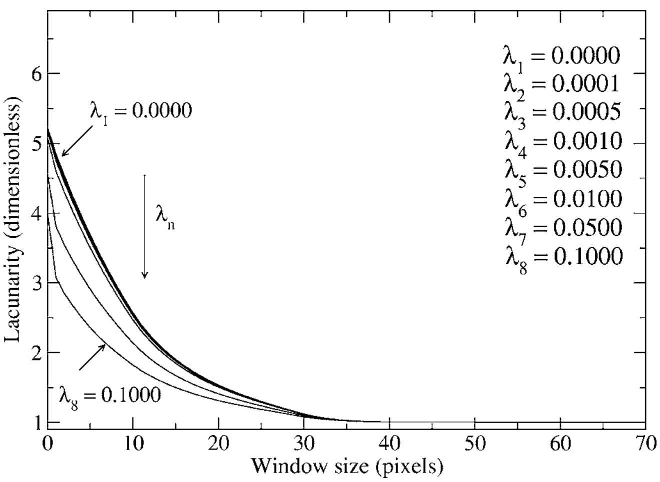

FIG. 9. The resilience of the standard lacunarity against perturbation by Poisson noise with varying density λ.

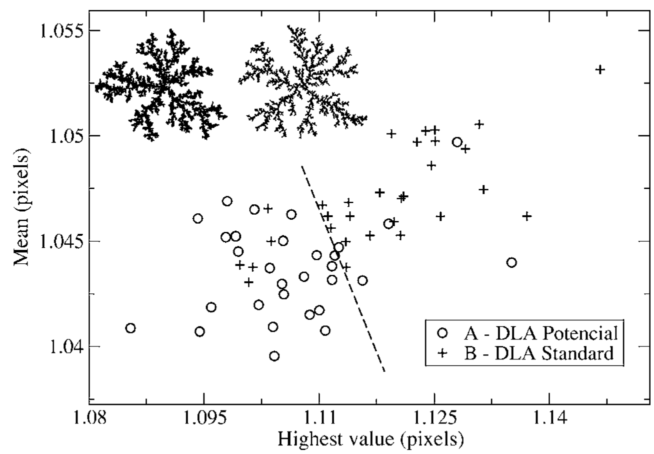

FIG. 11. A scatterplot defined by functionals extracted from the lacunarity curve, namely, the global mean and the local maximum values. The plot shows that the two types of DLA considered can be reasonable separated by the shown dashed line.

has been carried out in which two differently grown sets of DLA structures [14,15] with 30 samples each, are analyzed by the self-referred approach described in this paper. These two types of DLAs correspond to the standard DLA, already considered for the example in Fig. 8, and the *potential* DLAs [15]. The latter type of DLA is characterized by the incorporation of an equilibrium electric potential around the growing structure, which is obtained by solving the Laplace equation. The results of this experiment are summarized in Fig. 11, which show two clearly separated clusters (as illustrated by the straight dashed frontier), with some overlap at their borders. Such an overlap is a consequence of some degree of similarity between the two types of structures, which was detected by the considered measurement. It is observed that, out of the two considered measurements, the peak lacunarity value (represented along the x axis) contributed more effectively to the separation between the two classes of objects.

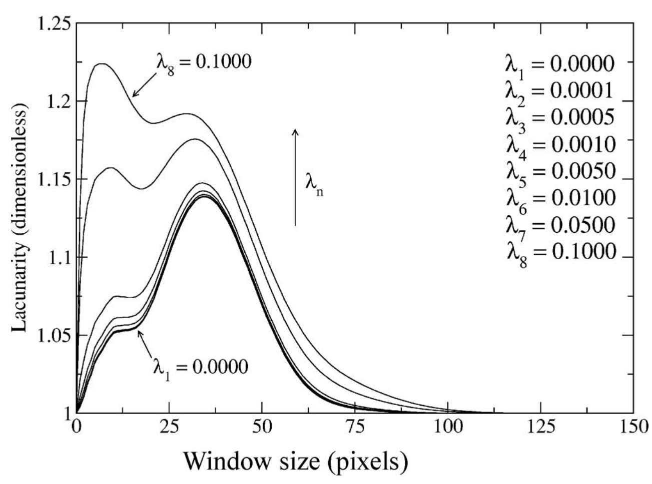

FIG. 10. The resilience of the proposed self-referred lacunarity against perturbation by Poisson noise with varying density λ.

III. COMMENTS AND CONCLUSIONS

The traditional approach to lacunarity estimation involves sliding a box throughout the space where the structure under analysis is contained, whereas the application of such a procedure implies substantial arbitrariness when applied to general shapes and sets of objects characterized by finite size. Of particular importance is the fact that there is no established criterion for defining the positions of the sliding box along the space under analysis, so that different implementations will often converge to different results. We have shown that the adoption of the objects under analysis as the reference for positioning of the sliding window provides not only a fully objective procedure for lacunarity estimation, but also enhances its potential for discriminating between different classes of patterns. Such effects have been demonstrated with respect to the important problem of DLA pattern formation and analysis. In addition, the stability of the self-referred approach has been investigated with respect to Poisson perturbations, suggesting improved robustness. Moreover, the enhanced signature provided by the object-referred framework considered in this article makes the choice of the window geometry an important issue. In particular, we have shown that circular (spherical) windows provide superior properties when used for self-referred lacunarity estimation by promoting the isotropy of the analysis. An additional advantage allowed by the considered lacunarity definition is its substantially reduced demand for computational resources. As the sliding window is constrained to the object under analysis, the total of integrations along the window is reduced from a large area around the object to its own area, which often imply savings of an order of magnitude. Such results suggest the self-referred lacunarity as an interesting methodology, especially when considered jointly with the fractal dimension, for the objective quantification of geometrical properties of a broad variety of objects in the most diverse theoretical and applied areas. Of particular interest is the characterization of biological cells, such as neurons (e.g., [16–18]), as well as organs (e.g., heart) and even the shape of individuals (e.g., viruses and bacteria). Applications to the characterization of astronomical structures such as galaxies (e.g., [19]), which incorporate patterns of voids, may also particularly benefit from the invariance properties of the self-refered lacunarity. It should be finally noted that the extension of this methodology to higher dimensions is straightforward, though at the cost of additional computational resources.

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